Bell Inequality Test

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Entanglement is one of the most fundamental aspects of quantum mechanics. It gives rise to phenomena that cannot be explained classically. In this work, we will demonstrate the non-locality of quantum mechanics through the violation of Bell's Inequality.

I. INTRODUCTION

The EPR paradox is a thought experiment developed by Albert **E**instein, Boris **P**odolsky, and Nathan **R**osen in their well-known 1935 paper [4]. It illustrates an inherent paradox in the formulation of quantum mechanics. Under the Copenhagen interpretation, two entangled particles are in an uncertain state until measured. The measurement of the state of a particle will project it into a specific and certain state; the wave function collapses. At the same time, the state of the other particle is also determined with certainty. This is classified as a paradox because it implies communication at speeds greater than the speed of light, which contradicts Einstein's theory of relativity. Einstein proposed that a realistic and complete formulation of quantum mechanics should include additional variables that determine quantum states completely. Bohm proposed a hidden variable formulation of quantum mechanics that satisfies the principle of locality [2]. In 1964, Bell developed a set of inequalities that are satisfied if systems are governed by local realism but are violated if systems are governed by the non-local principles of quantum mechanics [1]. These inequalities became known as the Bell Inequalities and have many different formulations. Several experiments were implemented to test the violation of the Bell Inequalities.

In this work, we will demonstrate the violation of the CHSH inequality using pairs of polarization-entangled photons and following the experimental procedure laid out in Kwiat et, al [5].

II. EXPERIMENT

A. Theory and Experimental Setup

1. Polarization-Entangled Photons

Spontaneous parametric down-conversion, or SPDC, is a non-linear optical phenomenon that consists of the transformation of a high energy photon into two polarization-entangled photons of lower energy obeying conservation of energy and momentum. We use SPDC in Type I Barium Borate crystals (BBO) to produce a pair of entangled photons. The pair is considered distinguishable because the photons travel in different directions along two diametrically opposite paths on a cone. The wavefunction of the pair of photons can be written as

$$|\Psi\rangle = \frac{|H,H\rangle + |V,V\rangle}{\sqrt{2}}$$

where $|V\rangle$ and $|H\rangle$ represent vertical polarization and horizontal polarization, respectively. We can measure the polarization of the photons with respect to an axis with arbitrary angle θ from the vertical. This measurements yields two outcomes:

$$|V_\theta\rangle = \cos\theta\,|V\rangle - \sin\theta\,|H\rangle$$
$$|H_\theta\rangle = \cos\theta\,|H\rangle + \sin\theta\,|V\rangle$$

We obtain the same entangled state in this basis as well.

$$|\Psi\rangle = \frac{|H_\theta, H_\theta\rangle + |V_\theta, V_\theta\rangle}{\sqrt{2}} \tag{1}$$

$$\tag{2}$$

In this experiment, we can measure the polarization of one down-converted photon, and determine the polarization of the other one with certainty. This experiment violates the principle of locality because, at the time of the measurement, the two photons may have a space-like separation.

2. BBO crystals and SPDC

When a photon passes through a BBO crystal, it has 1 in 10^{10} probability of being down-converted in the following way

$$|V\rangle \longrightarrow |H\rangle\,|H\rangle$$

while $|H\rangle$ polarized photons pass through the crystal unaffected. In order to obtain the superposition in equation 1, we use an additional BBO crystal at a perpendicular orientation. This ensures that a fraction of the $|H\rangle$ polarized photons is converted in this way

$$|H\rangle \longrightarrow |V\rangle\,|V\rangle$$

But because the $|H\rangle$ polarized photon travels a longer distance before being down-converted, the $|V\rangle\,|V\rangle$ pair carries a phase difference ϕ. Thus for a photon that is polarized along an arbitrary angle θ, we obtain the following wavefunction

$$|\Psi_{\text{pair}}\rangle = \cos\theta\,|H,H\rangle + e^{i\phi}\sin\theta\,|V,V\rangle \tag{3}$$

3. *Setup*

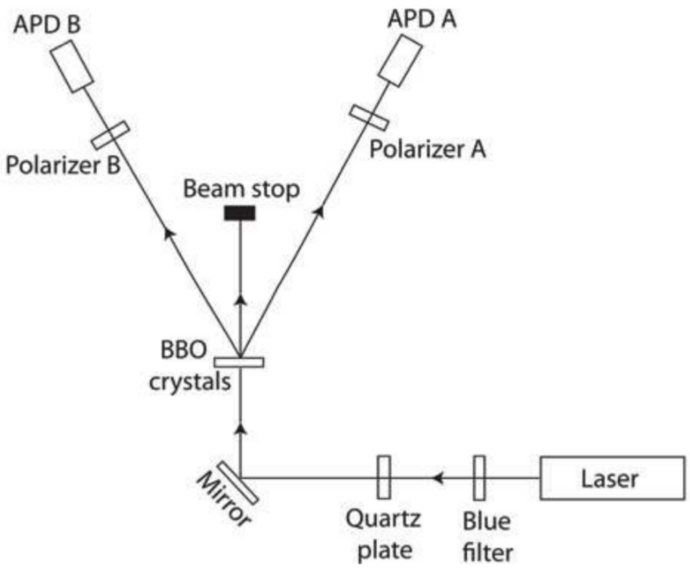

FIG. 1. Outline of the experimental setup (see [6]).

Figure 1 shows an outline of our experimental setup. We used a diode laser at 35 mW to produce high energy photons of wavelength 405.5 nm. The beam passes through a filter in order to remove unwanted wavelengths. The quartz plate is used to suppress the phase difference introduced at the BBO crystals (more on this in section II B 1). SPDC occurs at the BBO crystals that are mounted back-to-back with their optical axes at a right angle. The polarization-entangled photons travel on paths diametrically opposite on the down-conversion cone. Two avalanche photodiodes (APD) are used to detected the down-converted photons. The state of the photons is tuned using two polarizers placed in front the APDs.

The probability that the two photons from the down-converted pairs reach both ADPs is given by

$$\mathrm{P}(VV|\alpha,\beta) = |\langle V_\alpha, V_\beta|\Psi_{\text{pair}}\rangle|^2$$

where α and β are the angles of polarizer A and polarizer B, respectively. The other possible outcomes are

$$\mathrm{P}(HV|\alpha,\beta) = |\langle H_\alpha, V_\beta|\Psi_{\text{pair}}\rangle|^2$$
$$\mathrm{P}(VH|\alpha,\beta) = |\langle V_\alpha, H_\beta|\Psi_{\text{pair}}\rangle|^2$$
$$\mathrm{P}(HH|\alpha,\beta) = |\langle H_\alpha, H_\beta|\Psi_{\text{pair}}\rangle|^2$$

Writing out the probability $\mathrm{P}(VV|\alpha,\beta)$ using Eq.(3) gives

$$\begin{aligned}\mathrm{P}(VV|\alpha,\beta) \;&= \sin^2\alpha\,\sin^2\beta\,\cos^2\theta\,+\\ &\cos^2\alpha\,\cos^2\beta\,\sin^2\theta\,+\\ &\sin2\alpha\,\sin2\beta\,\sin2\theta\,\cos\phi\end{aligned}$$

Assuming a continuous flux of photons during the acquisition window, the number of coincidence pairs is

$$\begin{aligned}\mathrm{N}(VV|\alpha,\beta) \;=\; &A(\sin^2\alpha\,\sin^2\beta\,\cos^2\theta\,+\\ &\cos^2\alpha\,\cos^2\beta\,\sin^2\theta\,+\\ &\sin2\alpha\,\sin2\beta\,\sin2\theta\,\cos\phi) + B\end{aligned} \tag{4}$$

where A is the total number of pairs created and B is an offset due to random background photons. We can estimate these random coincidences with reasonable accuracy. Therefore, after subtraction of random coincidences, we can model the number of net coincidence photons using model $\mathcal{M}_0$:

$$\begin{aligned}\mathrm{N}(VV|\alpha,\beta) \;=\; &A(\sin^2\alpha\,\sin^2\beta\,\cos^2\theta\,+\\ &\cos^2\alpha\,\cos^2\beta\,\sin^2\theta\,+\\ &\sin2\alpha\,\sin2\beta\,\sin2\theta\,\cos\phi)\end{aligned} \tag{5}$$

The entangled state for which the CHSH inequality was formulated is a special case with $\theta = \frac{\pi}{4}$ and $\phi = 0$. The state of the down-converted photons becomes

$$|\Psi_{\text{pair}}\rangle \;=\; \frac{|H,H\rangle + |V,V\rangle}{\sqrt{2}} \tag{6}$$

and the probability of observing both photons at the APDs has a simple cosine squared dependence

$$\mathrm{P}(VV|\alpha,\beta) = \frac{1}{2}\cos^2(\alpha-\beta) \tag{7}$$

and

$$\mathrm{N}(VV|\alpha,\beta) = \frac{A}{2}\cos^2(\alpha-\beta) + B \tag{8}$$

B. Experimental Procedure

1. *Optimization*

The angle of polarization of the laser was optimized to ensure equal counts for $\alpha = 0$, $\beta = 0$ and $\alpha = 90$, $\beta = 90$, that is $\sin\theta = \cos\theta$. Then the vertical and horizontal axes the quartz plate were adjusted to maximize the number of counts for three sets of angles (0,0), (45,45), and (135,135) (see Fig.2.).

The optimal vertical angle of the quartz was determined to be $10.0° \pm 0.25°$. The

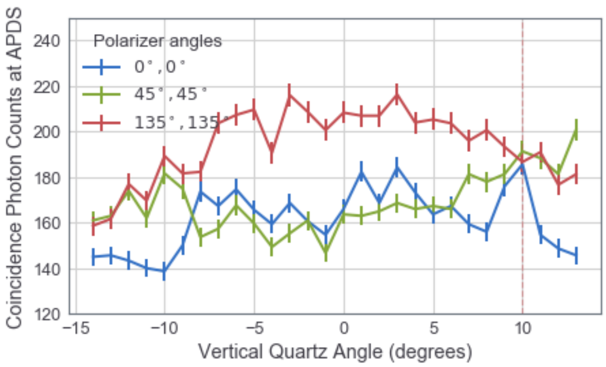

FIG. 2. Optimization of the vertical angle of the quartz plate to maximize the number of coincidence counts ($\phi = 0$). The best quartz angle is shown by the red dashed line at about $10.0° \pm 0.25°$ from the vertical.

2. *Counting DC photons*

Modeling the detection rate of coincidence photons is important in order to account for random coincidences of photons that are not a down-converted pair. The number of detected true DC pairs is given by

$$N_{\text{DC}} = N_{\text{coincidence}} - N_{\text{random}} \tag{9}$$

With this apparatus, detections within a $t_s = 26\,\text{ns}$ window are considered to be simultaneous. The labview program measures single photon counts (N_A, N_B) at each APD in $t_{acq} = 1\,\text{s}$ window. For a Poisson process the count rate is constant, and the expected number of counts at each APD within the the t_s window is

$$(N_A \frac{t_s}{t_{acq}}, N_B \frac{t_s}{t_{acq}})$$

The probability that a photon detection occurs in APD_1 within t_s of a detection in APD_2 is $N_A \frac{t_s}{t_{acq}}$, and the total number of such random coincidences is

$$N_{\text{random}} = N_A N_B \frac{t_s}{t_{acq}}$$

For each measurement in this experiment we count the number of coincidences $N_{\text{coincidence}}$ per 1s acquisition window a total of three times. We calculate the estimator of the distribution using a simple arithmetic mean $\bar{\mu} = \sum_i^3 \frac{N_{\text{coincidence}i}}{3}$ and we obtain the net coincidence count by $N_{\text{DC}} = \bar{\mu} - N_{\text{random}}$ with error $\sigma = \sqrt{N_{\text{DC}}}$.

3. $\cos^2$ *dependence*

Measurement of the cosine squared dependence of the number of DC photons is a good confirmation as to whether the DC photons are in fact in the state prescribed in Equation (6). The results are shown in Figure(3).

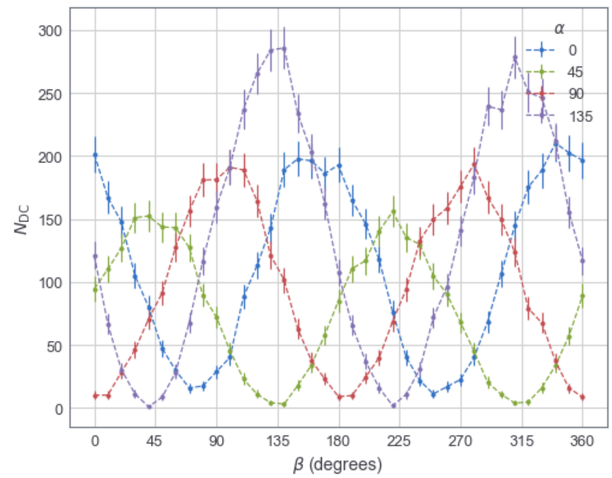

FIG. 3. Cosine squared dependence of the number of coincidence down-converted photons at four different angles of polarizer A.

Large discrepancies in the amplitudes of the different curves are apparent in Figure(3) and can be attributed to misalignment of the apparatus. However, Equation(5) cannot be used to fit this data (see Appendix B). Investigation of simulated data (see Appendix A) showed that the data collected has a systematic offset for the measured values of α and β by about 45°. Adding two new offset parameters produces a much better fit. The new model $\mathcal{M}$ is

$$\begin{aligned} \text{N}(VV|\alpha,\beta) \;=\; & A(\sin^2(\alpha+\alpha_0)\sin^2(\beta+\beta_0)\cos^2\theta + \\ & \cos^2(\alpha+\alpha_0)\cos^2(\beta+\beta_0)\sin^2\theta + \\ & \sin 2(\alpha+\alpha_0)\sin 2(\beta+\beta_0)\sin 2\theta\cos\phi) \end{aligned} \tag{10}$$

And the offsets were estimated to be $41.5° \pm 0.7°$ and $46.2° \pm 0.7°$ for α and β, respectively. The likelihood ratio of the original model $\mathcal{M}_0$ (Equation(5)) and the model with the additional offset parameters, $\mathcal{M}$ is

$$-2\text{Log}\frac{\mathcal{L}_0(\mathcal{M}_0)}{\mathcal{L}(\mathcal{M})} = 1316.7$$

which greatly favors the model with the offsets.

Taking this offset into account, we can estimate the remaining parameters of Equation(10). The results are shown in Figure(5).

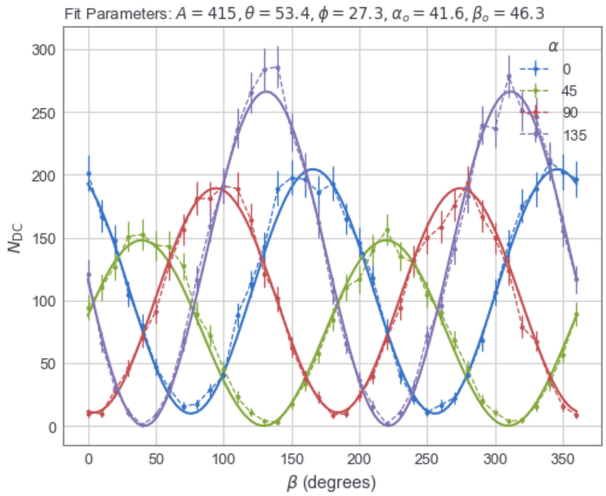

FIG. 4. Fit of Equation(5) for different polarizer angles with offset correction. The dashed lines represent the raw data while the solid lines represent the fit.

Fit parameters:

$$\begin{aligned} A &= 415 \pm 8 \\ \theta &= 53.3^\circ \pm 0.6^\circ \\ \phi &= 27.2^\circ \pm 2.7^\circ \\ \alpha_0 &= 41.5^\circ \pm 0.7^\circ \\ \beta_0 &= 46.2^\circ \pm 0.7^\circ \end{aligned}$$

Although our DC photons are not exactly in the state from Equation(6), violations of Bell's inequality are still possible as we will see in the next section.

4. *CHSH Inequality*

In Einstein's view, the experiment above can be described in terms of a "hidden variable" such that locality is preserved. Bell formulated an equality that constrains any such HVT but is violated by quantum mechanics. We will use the CHSH inequality as derived by John **C**lauser, Michael **H**orne, Abner **S**himony, and Richard **H**olt, who described it in a paper published in 1969 [3]. The proof uses the following quantity

$$S = E(a, b) + E(a, b') + E(a', b') - E(a', b) \tag{11}$$

where $E(\alpha, \beta)$ is defined as

$$\begin{aligned} E(\alpha, \beta) = &\big[\mathrm{P}(VV|\alpha, \beta) + \mathrm{P}(HH|\alpha, \beta)\big] - \\ &\big[\mathrm{P}(HV|\alpha, \beta) + \mathrm{P}(VH|\alpha, \beta)\big] \end{aligned} \tag{12}$$

The quantity S has little physical meaning. However, it satisfies

$$|S| \leq 2$$

for any HVT and arbitrary angles a, b, a', b' [3]. Quantum mechanics predicts otherwise. Indeed, for the set of angles $a = -45^\circ$, $a' = -0^\circ$, $b = -22.5^\circ$, $b' = 22.5^\circ$, quantum mechanics predicts

$$S_{QM} = 2\sqrt{2} \tag{13}$$

This value is specific to the rotationally invariant state of $|\Psi_{\text{pair}}\rangle$ (Equation 6). Other states have a lower value of S. We shall see in the results section that, even though we were not exactly in the state $|\Psi_{\text{pair}}\rangle$ we were able to obtain a significant violation of the CHSH inequality.

C. Results

The probabilities in Equation(12) are given by

$$\begin{aligned} \mathrm{P}(VV|\alpha, \beta) &= N(\alpha, \beta)/A \\ \mathrm{P}(HV|\alpha, \beta) &= N(\alpha + \frac{\pi}{2}, \beta)/A \\ \mathrm{P}(VH|\alpha, \beta) &= N(\alpha, \beta + \frac{\pi}{2})/A \\ \mathrm{P}(HH|\alpha, \beta) &= N(\alpha + \frac{\pi}{2}, \beta + \frac{\pi}{2})/A \end{aligned}$$

where A is the total number of coincidences $N(\alpha, \beta) + N(\alpha + \frac{\pi}{2}, \beta) + N(\alpha, \beta + \frac{\pi}{2}) + N(\alpha + \frac{\pi}{2}, \beta + \frac{\pi}{2})$. Using these probabilities we calculated S to be

$$S = 2.75 \pm 0.02$$

Where the uncertainty is simply

$$\sigma_S = \sqrt{\sum_i N_i (\frac{\partial S}{\partial N_i})^2}$$

This is a clear violation of the CHSH inequality by many standard deviations.

The values used for this calculation are reported in Table 1 in Appendix B.

III. CONCLUSION

Although our down-converted photons were not in the state with $S_{\text{QM}} = 2\sqrt{2}$, we were still able to obtain a statistically significant violation of Bell's Inequality and see evidence for non-local interactions between entangled particles. A more rigorous optimization of the apparatus will improve the measured value of S.

Appendix A: Simulation

The program that was used to generate simulated data can be found in the `sim.ipynb` file in the `/supplementary material/code` folder. Here are some plots of simulated data

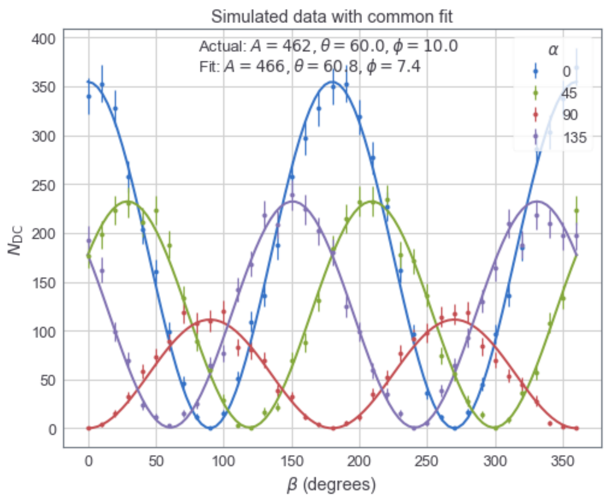

FIG. 5. Simulated data generated from random values of N, θ, ϕ. The fit is represented by the solid lines.

Fit parameters:

$$
\begin{aligned}
p &= \hat{p} \pm \sigma_p(\text{actual}) \\
A &= 466 \pm 9(460) \\
\theta &= 60.8^\circ \pm 0.6^\circ(60) \\
\phi &= 7.4^\circ \pm 5.5^\circ(10) \\
\alpha_0 &= -1.1^\circ \pm 0.6^\circ(0) \\
\beta_0 &= 0.7^\circ \pm 0.6^\circ(0)
\end{aligned}
$$

Appendix B: $\mathcal{M}_0$ Best Fit

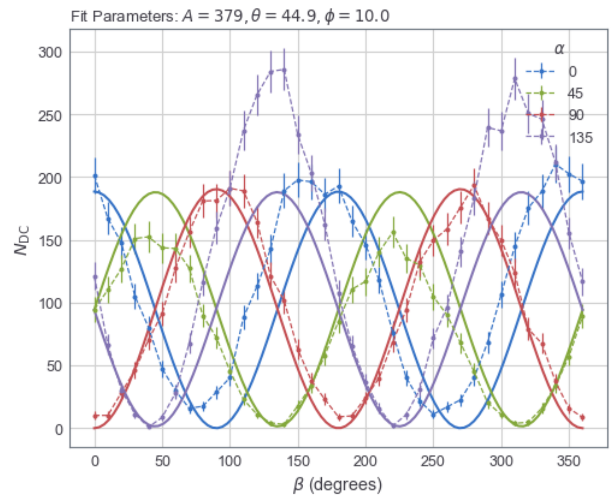

FIG. 6. Best fit of model $\mathcal{M}_0$. The dashed lines represent the raw data while the solid lines represent the fit.

The fit obtained using $\mathcal{M}_0$ is significantly different from the real data.

Appendix C: S Data

Polarizer A	Polarizer B	N_A	N_B	N_{random}	$N_{coincidence}1$	$N_{coincidence}2$	$N_{coincidence}3$	Average $N_{coincidence}$	N_{DC}
-45.0	-22.5	12859	13471	5	130	128	168	142	137
-45.0	22.5	12594	13418	4	14	13	7	11	7
-45.0	67.5	12995	13467	5	52	43	45	47	42
-45.0	112.5	13330	13372	5	195	150	192	179	174
0.0	-22.5	12563	13263	4	135	161	150	149	144
0.0	22.5	12638	13343	4	121	140	124	128	124
0.0	67.5	12781	13251	4	18	26	19	21	17
0.0	112.5	13290	13051	5	45	29	36	37	32
45.0	-22.5	12415	12844	4	27	31	34	31	27
45.0	22.5	10698	11774	3	145	162	162	156	153
45.0	67.5	11604	12174	4	165	167	166	166	162
45.0	112.5	11037	11613	3	43	28	32	34	31
90.0	-22.5	12705	12913	4	22	24	25	24	19
90.0	22.5	12657	13075	4	57	45	53	52	47
90.0	67.5	13004	13072	4	158	172	170	167	162
90.0	112.5	13364	13023	5	153	143	139	145	140

TABLE I. The measurements used to calculate the S value.

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